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A study of Duffing's equation

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and find that in form and content it is worthy to submit to the examination committee.

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I concur with the above approval

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A STUDY OF DUFFING'S EQUATION

BY

CHARLES KIRBY MURCH

A

THESIS

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ABSTRACT

The unforced cases of Duffing's equation and the equation of the simple pendulum are considered by imposing certain conditions on a linear partial differential equation. Geometrical arguments are presented which lead to solutions of the special cases considered.

Undetermined functions of integration in the solutions limit their use to the approximation of systems with small non-linearity.

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LIST OF SYMBOLS

A - Amplitude.

c - Coefficient of viscous damping term.

C - Constant of integration.

$C(\gamma)$ - Constant of integration which depends upon choice of separation parameter.

t - Time.

x - Displacement or angular displacement.

α - Positive coefficient of the linear term of the restoring force. Positive coefficient of the restoring force for the simple pendulum.

β - Coefficient of the non-linear term of the restoring force. May be either positive or negative.

γ - Separation parameter. Angular frequency when derived from separation parameter.

φ - Constant of integration.

$\varphi(\gamma)$ - Constant of integration which depends upon choice of separation parameter.

ω - Angular frequency.

Cn(u) - Elliptic consine of u.

$F(\lambda,\varphi)$ - Incomplete elliptic integral of the first kind with modulus λ.

$F(\lambda,\pi/2)$ - Complete elliptic integral of the first kind.

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I. INTRODUCTION

The equation

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + f(x) = F\cos\omega t \tag{1}$$

was first systematically studied by Duffing[1] in 1918 with $f(x) = \alpha x + \beta x^3$. c and α are positive constants and β may be either positive or negative. The equation has an exact solution only when c and F are zero. However, many methods of approximating a solution have been devised for this equation and the special cases when c or F or both pass to zero.

The system to which equation (1) refers is subject to viscous damping, an applied periodic force, and a non-linear restoring force f(x) which is an odd analytic function. If the mass were displaced an amount $x = A$ and released, the system would oscillate about the equilibrium position $x = 0$.

If β in equation (1) is made very small then the periodic motion of the system approximates that of a linear harmonic oscillator. Most methods of solution use the solution of the linear system as a first approximation and for this reason should yield acceptable results only when the system is 'quasi-linear', i.e. $\beta \ll \alpha$.[2]

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[1] Duffing, G., Erzwungene Schwingungen bei veranderlicher Eigenfrequenz.

[2] Minorsky, N., Introduction to Non-Linear Mechanics, p. 135

The experimental results of Ludeke[3] however indicate that when β is large the form of the solution differs only slightly from the quasi-linear system. Systems with large non-linearity still exhibit solutions having the appearance of slightly distorted sinusoidal functions.

Unlike linear systems, the motions described by equation (1) are not always isochronous. The frequency[4] of oscillation is dependent upon the amplitude A. An important part of any solution is the relationship between amplitude and frequency. As an example, for the undamped, free system the frequency-amplitude relation has the form $\omega^2 = \alpha + \xi\beta A^2$ where ξ is a parameter which depends upon the approximation method employed.

The purpose of this study is to determine a solution to the undamped, free system. Since this system has an exact solution, the results may be checked accurately. The method of solution is then applied to the more general damped, free case which has no known exact solution. In addition, the case of the simple pendulum is studied. The simple pendulum is a more general extension of the undamped, free form of Duffing's equation. No forced systems are investigated.

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(3) Ludeke, Carl A., J. App. Phy. 20, 694.

(4) The terms 'frequency' and 'angular frequency' will be used interchangeably when no confusion is likely to result.

II. REVIEW OF THE LITERATURE

A brief discussion of the methods of solution of the equations of interest follows.

1) The undamped, free form of Duffing's equation is written

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = 0. \tag{2}$$

A well known exact solution is expressed in the form of the 'inverse' function[5].

$$t = \frac{1}{(\alpha + \beta A^2)^{1/2}} \int_0^{\varphi} \frac{d\varphi}{(1 - \lambda^2 \sin^2 \varphi)^{1/2}} \tag{3}$$

where $\lambda^2 = \dfrac{\beta A^2}{2(\alpha + \beta A^2)}$ and $\varphi = \cos^{-1}\left(\dfrac{x}{A}\right)$.

The integral in the right side of equation (3) is an incomplete elliptic integral of the first kind with **modulus** λ .

In standard notation the solution is written

$$t = \frac{1}{(\alpha + \beta A^2)^{1/2}} F(\lambda, \varphi). \tag{4}$$

Values of $F(\lambda, \varphi)$ corresponding to given values of φ may be found in tables of elliptic integrals[6]. The solution has the form of an

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[5] McLachlan, N. W., Ordinary Non-Linear Differential Equations in Engineering and Physical Sciences. p. 27.

[6] Jahnke, Eugene and Fritz Emde, Tables of Functions with Formulae and Curves.

elliptic cosine $\mathrm{cn}(\omega t)$ which is shown graphically on figure 2. From equation (4) one may easily compute the angular frequency ω for given values of the parameters α, β, and A.

A widely used approximate method[7] of solving equation (2) employs as a first approximation the relation

$$x = A_1 \cos \omega t + A_3 \cos 3\omega t \qquad (A_3 << A_1).$$

Upon substitution into equation (2) one is led to the frequency-amplitude relation $\omega^2 = \alpha + \frac{3}{4}\beta A^2$.

2) The motion of the simple pendulum is described by the equation

$$\frac{d^2x}{dt^2} + \alpha \sin x = 0 \tag{5}$$

where $\alpha = g/l$ and x represents the angular displacement. The system may be solved by use of the elliptic integral[8].

$$t = -\alpha^{-\frac{1}{2}} \int_0^{\varphi} \frac{d\varphi}{(1-\lambda^2 \sin^2 \varphi)^{\frac{1}{2}}} \tag{6}$$

where $\lambda = \sin\left(\frac{A}{2}\right)$, $\varphi = \sin^{-1}\left(\frac{1}{\lambda \sin(x/2)}\right)$

Integration of equation (6) between the limits $\varphi = (\pi/2, 0)$ yields one-fourth the period τ_0 , therefore

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[7] McLachlan, N. W., Theory of Vibrations. p. 48.

[8] Davis, Harold T., Introduction to Non-Linear Differential and Integral Equations. p. 192.

$$T_0 = 4\left(\frac{l}{g}\right)^{1/2} F(\lambda, \tfrac{\pi}{2}) \tag{7}$$

is the time for a complete period. Clearly the motion is not isochronous; the angular frequency is a function of λ and therefore a function of amplitude.

Since in terms of a Maclaurins series $\sin x = x - x^3/3! + x^5/5! - \ldots$, equation (5) may be approximated by equation (2) provided the amplitude is kept fairly small ($A \not< 30°$). β becomes $-g/6l$ and the same methods of solution can be employed.

3) The free, damped case of Duffing's equation is given by

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \alpha x + \beta x^3 = 0. \tag{8}$$

No exact solution is known for this equation. However, if c is small the motion is nearly cosinusoidal with angular frequency

$$\omega \simeq (\alpha + \tfrac{3}{4}\beta A^2)^{1/2}$$

and equation (8) may be replaced by the approximate linear equivalent linear form[9]

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \omega^2 x = 0$$

which has the solution $x = Ae^{-\frac{c}{2}t}\cos\omega t$.

The coefficient of the cosine term now represents the amplitude,

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[9] McLachlan, N. W., Ordinary Non-Linear Differential and Integral Equations. p. 33.

so that the frequency amplitude relation may be written

$$\omega = \left(\alpha + \frac{3}{4}\beta A^2 e^{-ct}\right)^{1/2}$$

The difficulty now is that the coefficients in the equivalent equation are no longer constants, but are now variables in amplitude and time.

Intuitively, equation (8) represents an oscillation with both amplitude and frequency decreasing with time. The amplitude approaches zero with increasing time. Since the frequency squared is approximately a linear function of A^2 [(10)] the frequency approaches the limiting value $\alpha^{\frac{1}{2}}$.

[(10)] Ludeke, Carl A., J. App. Phy. 20, 694.

III. DISCUSSION

The method of solution of the special cases of interest of equation (1) makes use of a partial differential equation which has a known solution.

A second order linear partial differential equation with dependent variable x and independent variables y and t is chosen:

$$\frac{\partial^2 x}{\partial t^2} + c\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y} = 0 \tag{9}$$

The second order term in x and t corresponds to the first term in equation (1), the first order term in x and t corresponds to the viscous damping, and the remaining term in x and y corresponds to the restoring force.

The solution to the linear partial differential equation is a surface in x, y, t-space. The partial differential equation is related to Duffing's equation by the condition that the first order term in x and y is equal to the restoring force. The solution to this equation is then a second surface. Along the intersection of these surfaces, x is given implicitly as a function of time and Duffing's equation must be satisfied.

The following assumptions are made about the nature of the solutions of equation one. (a) A harmonic solution exists which is a function of the parameters α, β, A and γ. (b) The solution changes continously as any of the parameters pass to zero as a limit and the solution becomes the solution to the resulting differential equation.

(a) The undamped, free system with cubic restoring force is given by equation two.

The associated partial differential equation

$$\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial y} = 0 \tag{10}$$

has a general solution

$$x(y,t) = \int d\gamma\, C(\gamma) \exp(\gamma^2 y) \cos[\gamma t + \varphi(\gamma)]. \tag{11}$$

where the integral is taken over the range of separation parameter which is appropriate. $C(\gamma)$ and $\varphi(\gamma)$ are constants of integration which depend on the range of separation parameter. A particular solution of interest is

$$x(y,t) = C \exp(\gamma^2 y) \cos[\gamma t + \varphi] \tag{12}$$

where C and φ are constants of integration.

The partial differential equation is related to Duffing's equation by the condition

$$\bar{y} = \log\left(\frac{x^2}{\alpha + \beta x^2}\right)^{\frac{1}{2\alpha}} + f(t) \tag{13}$$

which is the general solution of

$$\frac{\partial x}{\partial y} = \alpha x + \beta x^3$$

$f(t)$ is an arbitrary function of time.

The solution, expression (12), is a surface in x, y, t-space; the condition (13) is a second surface. The intersection of the surfaces described by equations (12) and (13) is a curve

$$x(\bar{y},t) = C\exp(\gamma^2\bar{y})\cos(\gamma t+\varphi)$$

along which

$$\frac{\partial^2 x}{\partial t^2} + \alpha x + \beta x^3 = 0$$

Since x may be written as an implicit function of t along this curve

$$\frac{d^2 x}{dt^2} + \alpha x + \beta x^3 = 0$$

and

$$x(t) = C\exp[\gamma^2 f(t)]\left[\frac{x^2}{\alpha+\beta x^2}\right]^{\frac{\gamma^2}{2\alpha}}\cos(\gamma t+\varphi) \tag{14}$$

is a solution to Duffing's equation. f(t) remaihs to be determined. Equation (14) will be reducible to the linear case if

$$f(t) = [\beta g(t)] + \frac{1}{2\alpha}\log\left[\frac{\alpha+\beta A^2}{A^2\cos^2(\gamma t+\varphi)}\right] \tag{15}$$

where g(t) is now the arbitrary function of time. With this choice and the substitution C=A exp $-\beta\gamma^2 g(0)$

$$x(t) = A\frac{\exp[\beta\gamma^2 g(t)]}{\exp[\beta\gamma^2 g(0)]}\left[\frac{x^2}{A^2\cos^2(\gamma t+\varphi)}\,\frac{\alpha+\beta A^2}{\alpha+\beta x^2}\right]^{\frac{\gamma^2}{2\alpha}}\cos(\gamma t+\varphi) \tag{16}$$

and

$$X(0) = A\left[\frac{X^2}{A^2}\,\frac{\alpha+\beta A^2}{\alpha+\beta X^2}\right]^{\frac{\gamma^2}{2\alpha}}$$

is satisfied by x(0) = A, when $\varphi = 0$.

If the linear case is approached by letting β become zero

$$\lim_{\beta\to 0} X(t) = A\left[\frac{X^2}{A^2\cos^2\gamma t}\right]^{\frac{\gamma^2}{2\alpha}}\cos\gamma t$$

This expression reduces to the identity

$$\frac{X}{A\cos\gamma t} = \left[\frac{X}{A\cos\gamma t}\right]^{\frac{\gamma^2}{\alpha}}$$

where $\gamma^2 = \alpha$

Expression (16) may now be rewritten

$$\frac{X}{A}\left[\frac{\alpha+\beta X^2}{\alpha+\beta A^2}\right]^{\frac{\gamma^2}{2(\alpha-\gamma^2)}} = \frac{\exp\left[\frac{\beta\gamma^2\alpha}{\alpha-\gamma^2}\,g(t)\right]}{\exp\left[\frac{\beta\gamma^2\alpha}{\alpha-\gamma^2}\,g(0)\right]}\cos\gamma t \tag{17}$$

where g(t) remains undetermined. Since there is apparently no criterion available which one may use to determine g(t), one is forced to make a simplifying approximation in order to obtain numerical results. The only choice immediately available is to set g(t) = 1. Expression (16) now becomes

$$A\cos\gamma t = X\left[\frac{\alpha+\beta X^2}{\alpha+\beta A^2}\right]^{\frac{\gamma^2}{2(\alpha-\gamma^2)}} \tag{18}$$

In order for the solution (18) to be of use, a value for the angular frequency must be determined. A frequency-amplitude relation can be readily obtained from the solution (18). Differentiating equation (18) with respect to time yields

$$-A\gamma \sin \gamma t = \dot{x}\left[\frac{\alpha+\beta x^2}{\alpha+\beta A^2}\right]^{\frac{-\gamma^2}{2(\gamma^2-\alpha)}}\left[\frac{\gamma^2\alpha-\alpha^2-\alpha\beta x^2-\beta x^2\gamma^2}{(\gamma^2-\alpha)(\alpha+\beta x^2)}\right]$$

Making use of the identity $\sin^2 u + \cos^2 u = 1$ results in the relation

$$x^2+\dot{x}^2\left[\frac{\gamma^2\alpha-\alpha^2-\alpha\beta x^2-\beta x^2\gamma^2}{\gamma(\gamma^2-\alpha)(\alpha+\beta x^2)}\right]^2 = A^2\left[\frac{\alpha+\beta A^2}{\alpha+\beta x^2}\right]^{\frac{-\gamma^2}{\gamma^2-\alpha}} \tag{19}$$

The first integral of equation (2), after applying the initial conditions x = A, v = 0, when t = 0, yields the conditions

$$\dot{x}^2 = \alpha A^2 + \frac{1}{2}\beta A^4 \qquad \text{when} \qquad x = 0$$

Substituting the above conditions into expression (19) yields the frequency-amplitude relation

$$\left(\alpha A^2+\frac{1}{2}\beta A^4\right)\left[\frac{1}{\gamma}\right]^2 = A^2\left[1+\frac{\beta}{\alpha}A^2\right]^{\frac{-\gamma^2}{\gamma^2-\alpha}}$$

In a more useful form the above relation becomes

$$\log\left(\alpha A^2+\frac{1}{2}\beta A^4\right) = \log A^2\gamma^2 - \frac{\gamma^2}{\gamma^2-\alpha}\log\left[1+\frac{\beta}{\alpha}A^2\right] \tag{20}$$

When β is small, the second term on the right side of the equation becomes negligible, and the expression reduces to the relation found by several other approximation methods. When $\beta = 0$, the frequency reduces to $\alpha^{\frac{1}{2}}$, the frequency of the linear system.

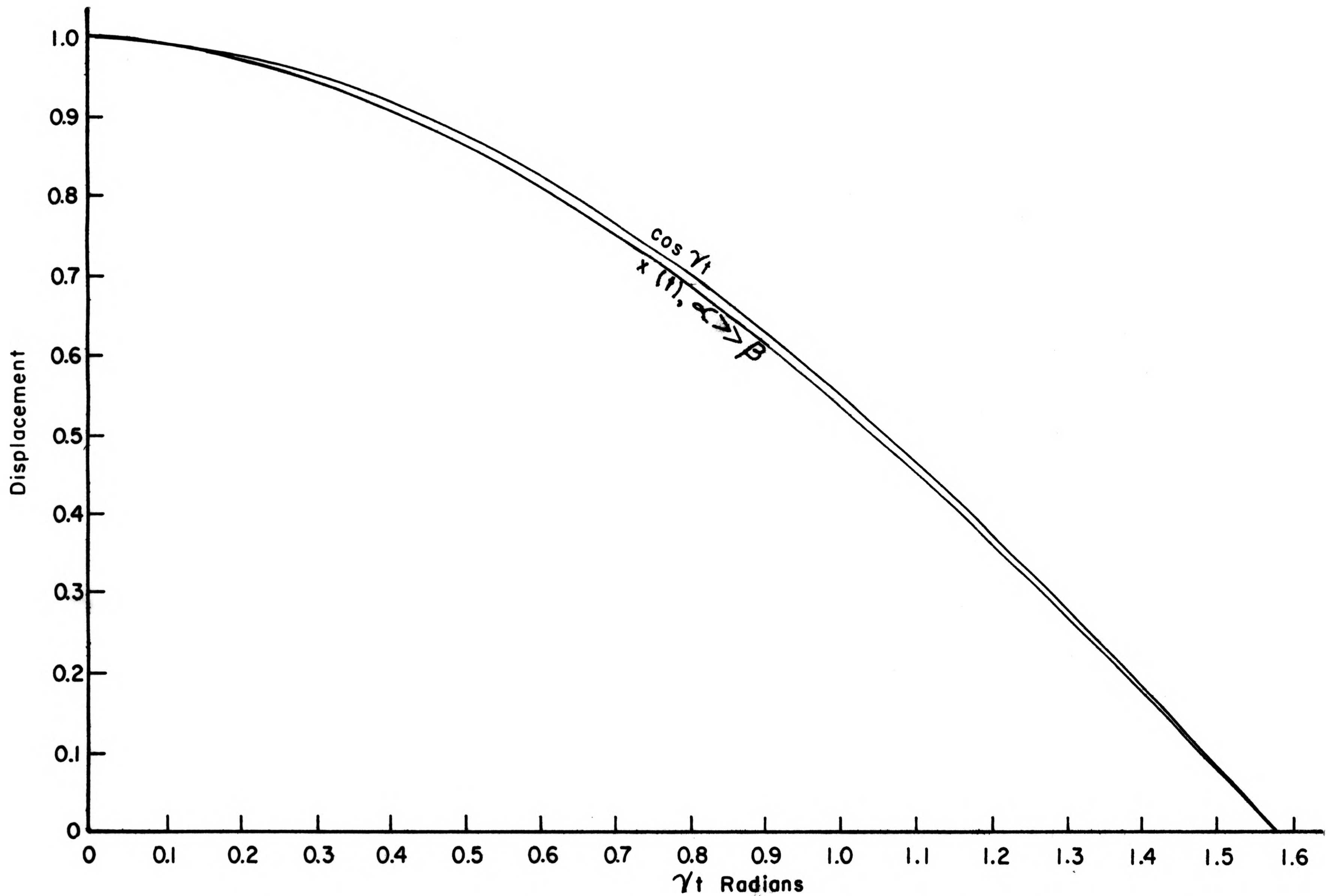

FIGURE NO. 1 - UNDAMPED FREE SYSTEM WITH SMALL NON-LINEARITY

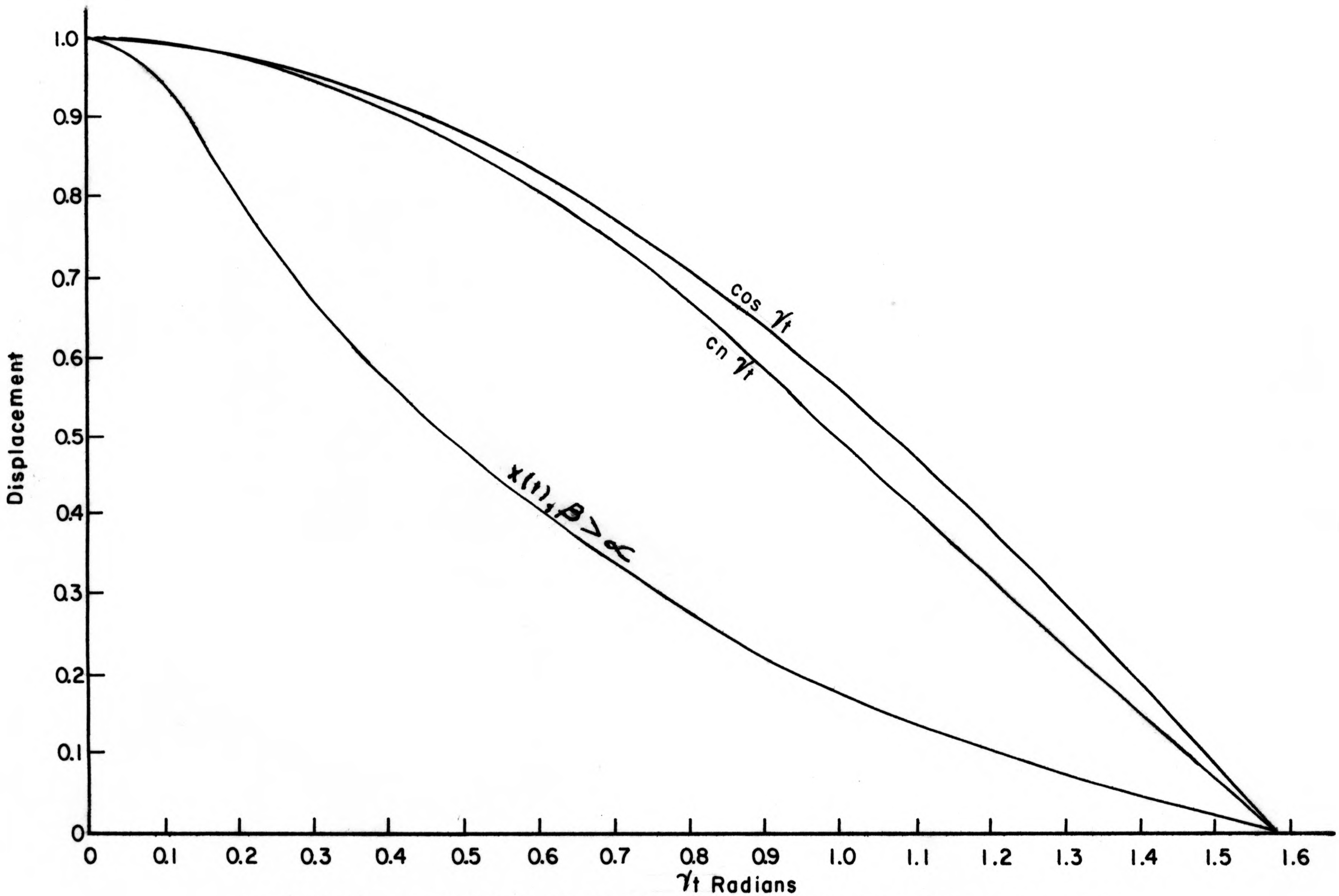

FIGURE NO. 2- UNDAMPED SYSTEM WITH LARGE NON-LINEARITY

Several numerical calculations were made from equation (18) for values of the parameters α and β corresponding to both large and small non-linearities.

A quasi-linear system was simulated by placing $\alpha = 334.1$, $\beta = 5$, and $A = 1$[11]. When these values are substituted into the frequency-amplitude relation, the relation is satisfied by $\gamma = 19.5$. The first quarter period of the solution is shown in figure 1. The solution to the linear system, a circular consine, is shown for comparison. The exact solution in terms of elliptic integrals lies between the two curves but so nearly coincides that it cannot be clearly shown on the figure.

That the solution yields a good approximation for a system with small non-linearity is a consequence of the assumption that the solution pass to that of the linear case when β becomes small.

A highly non-linear system was obtained by setting $\alpha = 10$, $\beta = 100$, and $A = 1$. These values result in an angular frequency $\gamma = 26.2$. Also shown are the circular cosine and the exact solution, an elliptic cosine of angular frequency $\gamma = 9.05$.

It is evident that the agreement with the known solution is poor for both the shape of the curve and the angular frequency. This indicates that the choice $g(t) = 1$ is inadequate when β is large.

(b) The simple pendulum is represented by equation (5). The associated partial differential equation and its general solution are

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[11]The values of the parameters are chosen for ease of calculation and to avoid interpolation in the elliptic integral tables.

given by equations (10) and (11). The particular solution of interest is equation (12). The partial differential equation is related to the equation of the simple pendulum by the condition

$$\bar{y} = \log\left(\tan\frac{x}{2}\right)^{\frac{1}{\alpha}} + f(\tau) \tag{21}$$

which is the general solution of

$$\frac{\partial x}{\partial y} = \alpha \sin x$$

Using the same geometrical reasoning as in the preceding section, the expression

$$x(t) = C \exp\left(\gamma^2 f(t)\right)\left[\tan\frac{x}{2}\right]^{\frac{\gamma^2}{\alpha}} \cos(\gamma t + \varphi) \tag{22}$$

is a solution to the equation of the simple pendulum. Equation (22) will be reducible to the linear case if

$$f(\tau) = \left[(\gamma^2 - \alpha)\, g(t)\right] + \frac{1}{\alpha} \log\left[\frac{1}{\tan\frac{A}{2} \cos(\gamma t + \varphi)}\right]$$

and the integration constant C is given the value

$$C = A \exp\left(-\gamma^2(\gamma^2 - \alpha)\, g(0)\right).$$

With this choice

$$x(t) = A \frac{\exp\left[\gamma^2(\gamma^2 - \alpha) g(t)\right]}{\exp\left[\gamma^2(\gamma^2 - \alpha) g(0)\right]} \left[\frac{\tan\frac{x}{2}}{\tan\frac{A}{2} \cos(\gamma t + \varphi)}\right]^{\frac{\gamma^2}{\alpha}} \cos(\gamma t + \varphi) \tag{23}$$

and

$$X(0) = A\left[\frac{\tan\frac{X}{2}}{\tan\frac{A}{2}}\right]^{\frac{\gamma^2}{\alpha}}$$

is satisfied by x(0) = A, when $\varphi = 0$.

Now if the linear case is approached by letting the angular displacement become small, then tan A/2 $\rightarrow$ A/2 and tan x/2 $\rightarrow$ x/2. Expression (23) now reduces to

$$\frac{X}{A\cos\gamma t} = \left[\frac{X}{A\cos\gamma t}\right]^{\frac{\gamma^2}{\alpha}}$$

which is an identity provided

$$\gamma^2 = \alpha$$

Expression (23) may be rewritten

$$\left(\frac{X}{A}\right)^{\frac{\alpha}{\alpha-\gamma^2}} \frac{\exp(\gamma^2\alpha g(t))}{\exp(\gamma^2\alpha g(0))}\left[\frac{\tan\frac{X}{2}}{\tan\frac{A}{2}}\right]^{\frac{\gamma^2}{\gamma^2-\alpha}} = \cos\gamma t \tag{24}$$

where g(t) again remains undetermined. If the undetermined function is now set equal to one, equation (24) becomes

$$\cos\gamma t = \left(\frac{X}{A}\right)^{\frac{\alpha}{\alpha-\gamma^2}}\left[\frac{\tan\frac{X}{2}}{\tan\frac{A}{2}}\right]^{\frac{\gamma^2}{\gamma^2-\alpha}} \tag{25}$$

A frequency-amplitude relation may be found using the same method as in the preceding section.

Several numerical calculations were made to test the validity of the approximation. Figure 3 shows the results of a calculation for a system with large amplitude. In the case of the simple pendulum large amplitude corresponds to large non-linearity since the approximation $\sin x \approx x$ is no longer valid. In this calculation $A = 40^\circ$ and $\alpha = 10$. The curve labeled $cn(\gamma t)$ is the exact solution obtained by direct integration. This elliptic cosine function has an angular frequency of $\gamma = \underline{3.06}$. A circular cosine curve is included for comparison since it is the solution to the linear system and a widely used approximation. The remaining two curves are solutions obtained from equation (25) using slightly different frequencies. The lower curve has the exact frequency obtained from expression (7). The shape of the curve may be improved considerably by sacrificing a small amount of accuracy in angular frequency. Such a curve is labeled $\gamma = 2.97$.

As before the approximation does not yield good results for systems with large non-linearity (amplitude). Once again this exhibits the fact that g(t) must be known to produce an acceptable solution.

It is interesting to note the close agreement between the exact solution and the circular cosine. The approximate solution $x = \cos\omega t$ is quite good even for large amplitude ($A < 90^\circ$) when the relation $\sin x \approx x$ is far from valid. The accuracy of this 'small amplitude' solution can be improved still more by using the frequency obtained from expression seven.

(c) The damped, free form of Duffing's equation is given by expression eight.

The associated partial differential equation, expression 9, has the general solution

$$x(y,t) = \int d\gamma\, C(\gamma)\exp(-\delta t)\exp(\gamma^2 y)\cos(\gamma t+\varphi)$$

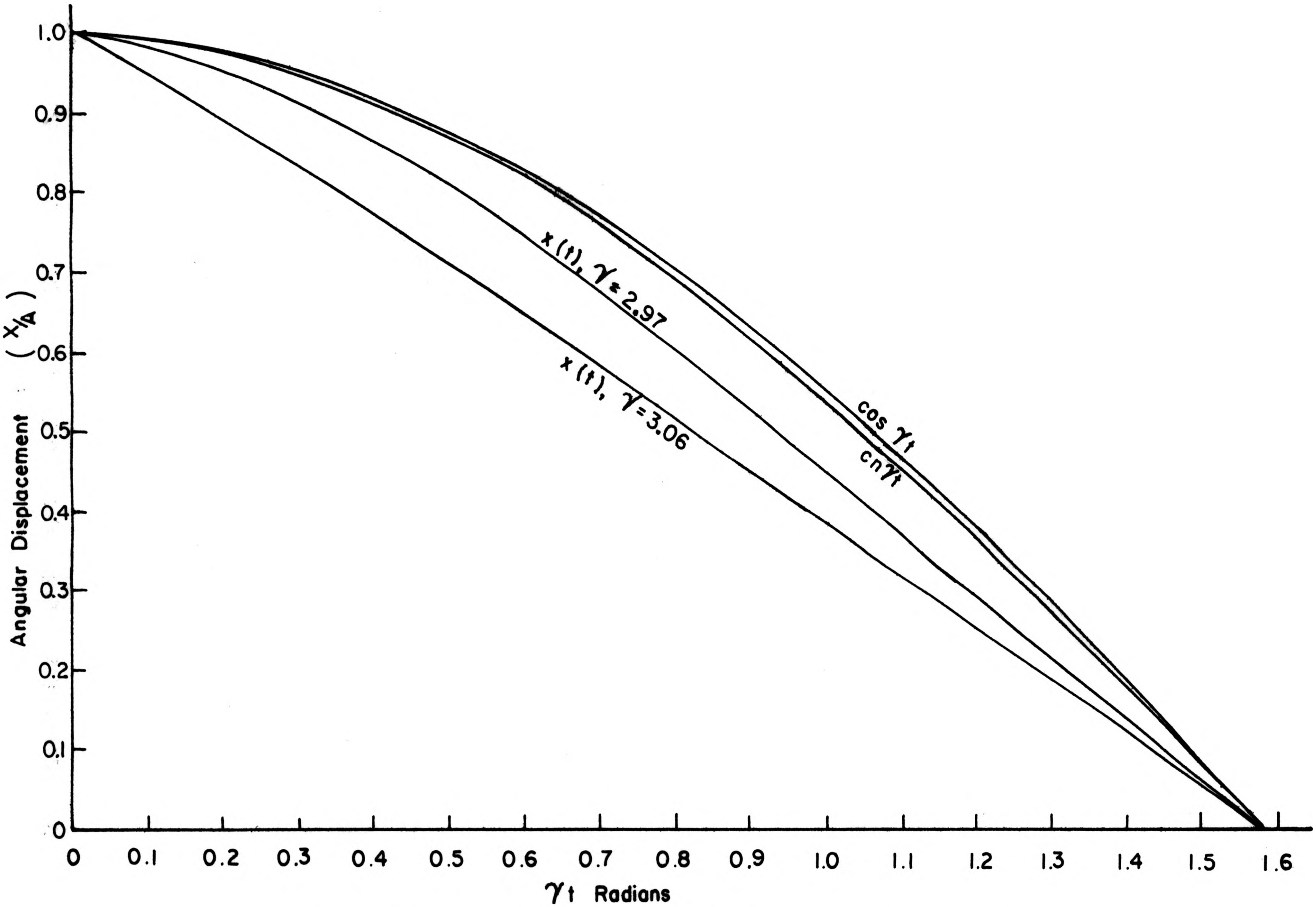

FIGURE NO. 3 - SIMPLE PENDULUM WITH LARGE AMPLITUDE where $\xi = \frac{c}{2}$. A particular solution of interest is

$$x(y,t) = C\exp(\gamma^2 y)\exp(-\xi t)\cos(\gamma t + \varphi).$$

The partial differential equation is related to equation (8) by the condition (13). Using the same geometrical reasoning as in the previous cases results in the expression

$$x(t) = C\exp(\gamma^2 f(t))\exp(-\xi t)\left[\frac{x}{\alpha + \beta x^2}\right]^{\frac{\gamma^2}{2\alpha}}\cos(\gamma t + \varphi) \tag{26}$$

which is a solution to equation (8). Equation (26) will be reducible to the proper form if

$$f(t) = \beta g(t) + \frac{1}{2\alpha}\log\left[\frac{\alpha + \beta A^2}{A^2\cos^2(\gamma t + \varphi)}\right]$$

and

$$C = A\exp(-\beta\gamma^2 g(0)).$$

Substituting this expression into the solution yields

$$x(t) = A\frac{\exp(\gamma^2\beta g(t))}{\exp(\gamma^2\beta g(0))}\exp(-\xi t)\left[\frac{x^2}{A^2\cos^2(\gamma t + \varphi)}\,\frac{\alpha + \beta A^2}{\alpha + \beta x^2}\right]^{\frac{\gamma^2}{2\alpha}}\cos(\gamma t + \varphi) \tag{27}$$

As β passes to zero as a limit, the exponentials (including the arbitrary functions of time) vanish and the solution passes to the solution of the damped, free linear oscillator. Furthermore, the solution is satisfied by $x(0) = A$ provided $g(t) = 1$ and $\varphi = 0$. Expression (27) may now be rewritten

$$\frac{X}{A}\left[\frac{\alpha+\beta X^2}{\alpha+\beta A^2}\right]^{\frac{\gamma^2}{2(\alpha-\gamma^2)}} = \exp\left(\frac{-\alpha\xi}{\alpha-\gamma^2}t\right)\cos\gamma t$$

The solution can be expected to yield good results provided β is small. Once again the undetermined function of time prohibits the use of the solution for systems with large non-linearities.

IV. CONCLUSIONS

General solutions to the three special cases of interest of Duffing's equation have been found. These solutions are intuitively acceptable and obey the assumptions made at the outset.

The obvious shortcoming of the investigation is the inability to resolve the undetermined functions of time included in each solution. However, since this function of time appears only in an exponent preceded by a coefficient which vanishes as the system approaches linearity, the solutions are valid approximations to the corresponding quasi-linear systems.

In general, the approximate solutions obtained yield no better results than several other approximation methods, even for systems with small non-linearity. The solutions are more cumbersome than those resulting from some other methods.

The method of solution appears to be valid and merits further study. A continuation of this investigation should disclose whether or not g(t) may be determined and exact solutions may be found.

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Note: The references marked with an asterisk(*) contain exhaustive bibliographies on non-linear mechanics and related fields.

VITA

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